B.Sc Physics Part-I, Paper-I, Group-B

Linear momentum Conservation:—

Lagrangian L in terms of generalized coordinates q_k and generalized velocity $\dot{q}_k$ is

$$L(q,\dot{q}) = L(q_1, q_2, \cdots, q_n, \dot{q}_1, \dot{q}_2, \cdots \dot{q}_n)$$

We consider here the Lagrangian of a closed system in an inertial frame and due to the homogeneous property of the inertial frame the Lagrangian will be invariant. Therefore, the variation in generalized coordinates will be zero

$$\delta L = \sum_k \frac{\partial L}{\partial q_k} \delta q_k + \sum_k \frac{\delta L}{\partial \dot{q}_k} \delta \dot{q}_k = 0 \quad \text{—— (1)}$$

δq_k → not function of t.

$$\therefore \quad \delta \dot{q}_k = \delta\left(\frac{dq_k}{dt}\right) = \frac{d}{dt}(\delta q_k) = 0 \quad \text{—— (2)}$$

from (1) and (2)

$$\delta L = \sum_k \frac{\partial L}{\partial q_k} \delta q_k = 0$$

Now for the arbitrary displacement δq_k, δL will be zero if

$$\frac{\partial L}{\partial q_k} = 0 \quad — \quad (3)$$

Since the Lagrange's equations for a conservation system is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0.$$

From (3),

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = 0$$

or $$\frac{\partial L}{\partial \dot{q}_k} = \text{constant} \quad — \quad (4)$$

We know that $L = T(\dot{q}_k) - V(q_k)$, we can write

$$\frac{\partial L}{\partial \dot{q}_k} = \frac{\partial}{\partial \dot{q}_k}(T - V)$$

$$= \frac{\partial}{\partial \dot{q}_k}(T) - 0$$

$$= \frac{\partial}{\partial \dot{q}_k}\left(\frac{1}{2} m \sum_k \dot{q}_k^2\right)$$

$$= m\dot{q}_k$$

$$= p_k$$

~~or from (4)~~ From (4)

$$\boxed{\frac{\partial L}{\partial \dot{q}_k} = p_k = \text{Constant}} \quad — \quad (5)$$

Therefore, for homogeneous space, the linear momentum of a closed system is constant or conserved.